A Short Introduction to Control Theory

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These are my notes to a short course on control theory I gave in February 2014 in the Dynamics Seminar of the Courant Institute of Mathematical Sciences. The topics are very much biased toward my own research interests. Nevertheless, I hope that some of the central ideas in control theory are conveyed.

Contents

1 Control Systems

Mathematical control theory studies models of all kinds of machines. A machine is a dynamical system whose behavior can be influenced by an external force in order to achieve a desired goal ("control task"). A standard example in control theory is the inverted pendulum[1], where the goal is the stabilization of an unstable equilibrium (the pendulum in the upright position). In the following, we want to look at the general structure of control systems. Restricting ourselves to deterministic models, a control system is usually described by difference or

[1] See `http://en.wikipedia.org/wiki/Inverted_pendulum`

differential equations of the form[2]

$$x_{t+1} = f(x_t, u_t) \quad \text{or} \quad \dot{x}(t) = f(x(t), u(t)),$$

depending on the model of time (discrete or continuous). Here x_t or $x(t)$ is the state of the system at time t that lives in a state space X (in continuous time, a differentiable manifold). In general, X could be infinite-dimensional. Then the above ODE could be a PDE with the spatial derivatives hidden in the definition of f. The external force is modelled by the function u (in discrete time, a sequence) which can be chosen from a given set $\mathcal{U}$ of functions and plugged into the right-hand side of the equation in order to produce the desired trajectories. Assuming that for all initial values in X the above equations have unique and globally defined solutions, we obtain a map

$$\varphi : \mathbb{T} \times X \times \mathcal{U} \to X, \quad (t, x, u) \mapsto \varphi(t, x, u), \qquad \mathbb{T} \in \{\mathbb{N}_0, \mathbb{Z}, \mathbb{R}_0^+, \mathbb{R}\},$$

called the *transition map* of the system, such that $\varphi(\cdot, x, u)$ is the solution for the initial value $x \in X$ at time zero and $u \in \mathcal{U}$ the control.

With the additional assumption that the set $\mathcal{U}$ is shift-invariant, i.e., that $u(\cdot + t) \in \mathcal{U}$ if $u \in \mathcal{U}$ and $t \in \mathbb{T}$, the transition map is a cocycle over the shift

$$\theta : \mathbb{T} \times \mathcal{U} \to \mathcal{U}, \quad (t, u) \mapsto \theta_t u = u(\cdot + t).$$

This means that the following identity is satisfied (as an easy implication of the uniqueness of solutions):

$$\varphi(t + s, x, u) \equiv \varphi(t, \varphi(s, x, u), \theta_s u).$$

Hence, we have a skew product system on the extended state space $\mathcal{U} \times X$,

$$\Phi_t(u, x) = (\theta_t u, \varphi(t, x, u)),$$

called the *control flow* of the system. We will come back to this dynamical point of view in Section 4.

2 Controllability: Linear Systems

One of the basic questions in control theory is whether a given system is *controllable*, i.e., whether it is possible to find for two given states $x, y \in X$ a control u such that $\varphi(t, x, u) = y$ for a time $t > 0$. If such u exists for an arbitrary choice of x and y, we call the system *completely controllable*. Intuitively it is clear that complete controllability almost never holds, in particular, if the system is

[2]A system in this form is also called an *open-loop* system. In contrast, in a *closed-loop or feedback system* the control is not merely a time-dependent function, but a function $u(x(t))$, depending on the state of the system. Not every control task that can be solved with open-loop controls can also be solved with a feedback, but in order to analyze the controllability properties, it makes sense to look at the open-loop system.

nonlinear and/or if there are restrictions on the functions in $\mathcal{U}$, as for instance boundedness. As a trivial example, consider the system

$$x_{t+1} = f(x_t, u_t) = x_t + u_t, \qquad x_t \in \mathbb{R},\ u_t \in (0,\infty).$$

Here it is not possible to reach states $x < x_0$ from a given initial state $x_0 \in \mathbb{R}$.

However, within the class of linear systems with no restrictions on the controls (except for technical ones) it makes sense to characterize the completely controllable ones, if only to transfer the result locally to nonlinear systems. For simplicity, let us consider continuous-time systems only:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad u \in \mathcal{U} := L^\infty(\mathbb{R}, \mathbb{R}^m). \tag{1}$$

Here $x(t) \in \mathbb{R}^d$ and $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{d\times m}$. Alternatively, we could consider locally integrable, locally essentially bounded, piecewise continuous or piecewise constant controls. The crucial point is that in control theory one does not want to be restricted to continuous controls, in order to model abrupt changes of the external force (like flipping a switch).[3] For measurable dependence of the right-hand side on t the theory of *Carathéodory differential equations* applies. Here the solutions are only (locally) absolutely continuous curves which satisfy the ODE almost everywhere (alternatively: continuous curves which satisfy the associated integral equation).

By the variation-of-constants formula the transition map of (1) is given by

$$\varphi(t,x,u) = \mathrm{e}^{At}x + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s.$$

For the question of complete controllability, note that by linearity it is sufficient to consider the set

$$\mathcal{R} := \bigcup_{t\geq 0} \mathcal{R}(t), \quad \mathcal{R}(t) := \{\varphi(t,0,u) : u \in \mathcal{U}\},$$

of states reachable from $x = 0$, which is a linear subspace. In fact, it is clear that $\mathcal{R}(t)$ is a linear subspace for each $t \geq 0$ and it holds that $\mathcal{R}(t) \subset \mathcal{R}(s)$ for $t < s$ (if $x \in \mathbb{R}^d$ can be reached from $x_0 = 0$ with the control u in time t, then it also can be reached in time s by the control v defined as $v(\tau) := 0$ for $\tau \in [0, s-t]$ and $v(\tau) := u(\tau - s + t)$ for $\tau \in (s-t, s]$.) Hence, for $y_1, y_2 \in \mathcal{R}$ we find t with $y_1, y_2 \in \mathcal{R}(t)$ and thus $y_1 + y_2 \in \mathcal{R}(t) \subset \mathcal{R}$. We want to show that

$$\mathcal{R} = \langle A|\mathrm{im}B\rangle := \mathrm{im}B + A\mathrm{im}B + A^2\mathrm{im}B + \cdots + A^{d-1}\mathrm{im}B.$$

By the theorem of Caley-Hamilton this is the smallest A-invariant subspace containing $\mathrm{im}B$. Another way to write it is

$$\langle A|\mathrm{im}B\rangle = \mathrm{im}W, \quad W := [B|AB|A^2B|\cdots|A^{d-1}B] \in \mathbb{R}^{d\times md}.$$

[3]See, e.g., `http://en.wikipedia.org/wiki/Bang-bang_control`

If $x \in \mathcal{R}$, then $x = \varphi(t, 0, u)$ for some $t \geq 0$ and $u \in \mathcal{U}$. For all $s \in [0, t]$ we have

$$\mathrm{e}^{A(t-s)}Bu(s) = \sum_{k=0}^{\infty} \frac{1}{k!}(t-s)^k A^k Bu(s) \in \langle A|\mathrm{im}B\rangle$$

and hence also

$$x = \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\mathrm{d}s \in \langle A|\mathrm{im}B\rangle.$$

The other inclusion is just a little more complicated. Let

$$W_t := \int_0^t \mathrm{e}^{As}BB^T\mathrm{e}^{A^Ts}\mathrm{d}s, \quad t > 0.$$

We prove that $\langle A|\mathrm{im}B\rangle \subset \mathrm{im}W_t$ or equivalently $[\mathrm{im}W_t]^\perp \subset \langle A|\mathrm{im}B\rangle^\perp$. If $x \in [\mathrm{im}W_t]^\perp$, then

$$0 = x^T W_t x = \int_0^t \|B^T\mathrm{e}^{A^Ts}x\|\mathrm{d}s$$

and hence $x^T\mathrm{e}^{As}B = 0$ for all $s \in [0, t]$, in particular $x^T B = 0$. Successive differentiating and evaluating in $s = 0$ gives $x^T A^k B = 0$ for all k. Hence, $x \in \langle A|\mathrm{im}B\rangle^\perp$. Now let $x \in \langle A|\mathrm{im}B\rangle$. Then there exists z with $x = W_t z$. With the control function

$$u(s) := B^T\mathrm{e}^{A^T(t-s)}z, \quad s \in [0, t],$$

we obtain

$$\begin{aligned}\varphi(t, 0, u) &= \int_0^t \mathrm{e}^{A(t-s)}BB^T\mathrm{e}^{A^T(t-s)}z\mathrm{d}s \\ &= \left(\int_0^t \mathrm{e}^{A\tau}BB^T\mathrm{e}^{A^T\tau}\mathrm{d}\tau\right)z = W_t z = x,\end{aligned}$$

and therefore $x \in \mathcal{R}$. Hence, we have found the equivalence

$$\mathcal{R} = \mathbb{R}^d \quad \Leftrightarrow \quad \mathrm{rk}[B|AB|A^2B|\cdots|A^{d-1}B] = d.$$

Since the set of points that can be steered to zero is also equal to $\langle A|\mathrm{im}B\rangle$, this yields a criterion for complete controllability, known as *Kalman's rank condition.*

Another important problem in control theory is the problem of stabilizability. Here the question is whether a limit set (e.g., an equilibrium) of an uncontrolled system (i.e., a classical dynamical system) can be stabilized by adding control terms to the equation. In the linear case, where $\dot{x}(t) = Ax(t)$ is the uncontrolled system with the equilibrium $x_0 = 0$, stabilizability means that for every $x \in \mathbb{R}^d$ there exists $u \in \mathcal{U}$ with $\varphi(t, x, u) \to 0$ for $t \to \infty$. Stabilizability holds iff the restriction of the system to the unstable subspace is controllable. In general, the state space $\mathbb{R}^d$ can be decomposed into a controllable subspace (which is $\langle A|\mathrm{im}B\rangle$) and an uncontrollable subspace. Then the system is stabilizable iff the eigenvalues of the uncontrollable part are stable. In this case, a matrix $F \in \mathbb{R}^{m\times d}$ exists such that $A+BF$ has only stable eigenvalues, i.e., the feedback system $\dot{x}(t) = (A+BF)x(t)$ is stable. However, notice that if a nonlinear system is stabilizable with open-loop controls, then in general it is not stabilizable by feedback.

If we restrict the set $\mathcal{U}$ of controls by imposing a condition like $u(t) \in U$ for a bounded set $U \subset \mathbb{R}^m$ with $\mathrm{int}U \neq \emptyset$ (which with regard to real-world systems of course is a more realistic assumption), we expect that the set of points reachable from 0 will not be a subspace anymore, but will still have nonempty interior if Kalman's rank condition holds (see Section 4).

3 Accessibility: Nonlinear Systems

Despite of many singular results, no general conditions for complete controllability of nonlinear systems are known. However, for a nonlinear system on a d-dimensional smooth manifold M with

$$\mathcal{U} = \{u \in L^{\infty}(\mathbb{R}, \mathbb{R}^m) \ : \ u(t) \in U \text{ a.e.}\}, \quad U \subset \mathbb{R}^m,$$

and right-hand side $f : M \times \mathbb{R}^m \to TM$ (sufficiently smooth) we can look at the finite-time orbits

$$\begin{aligned} \mathcal{O}^+_{\leq T}(x) &= \{y \in M \mid \exists t \in [0,T],\ u \in \mathcal{U} :\ y = \varphi(t,x,u)\}, \\ \mathcal{O}^-_{\leq T}(x) &= \{y \in M \mid \exists t \in [0,T],\ u \in \mathcal{U} :\ x = \varphi(t,y,u)\}, \end{aligned}$$

and ask for a condition for them to have nonempty interiors.

We call the system *locally accessible at x* if for every $T > 0$ both of these sets have nonempty interiors, and *locally accessible* if this happens for all $x \in M$.

The basic idea to find a condition for local accessibility in terms of the right-hand side of the system, is to look at maps of the form

$$\alpha : (t_1, \dots, t_d) \mapsto \varphi_{t_d,u_d} \circ \cdots \circ \varphi_{t_1,u_1}(x), \quad [0,T)^d \to M,$$

with $u_1, \dots, u_d \in U$ fixed. If this map is continuously differentiable and the derivative at $(t_1^*, \dots, t_d^*)$ has full rank, then the inverse function theorem guarantees that one can reach an open set around $\alpha(t_1^*, \dots, t_d^*)$ from x. In analyzing the derivative of such maps, the Lie brackets of the vector fields in

$$\mathcal{F} := \{f(\cdot, u) \ : \ u \in U\}$$

become involved. In fact, the iterated Lie brackets of elements in $\mathcal{F}$ give the possible directions in which one can steer the system from a given state.[4] Since

[4] This has to be made more precise. It is only true if one is allowed to combine trajectories of the system and trajectories of the time-reversed system.

every time we compute a Lie bracket, we lose one degree of differentiability, it makes sense to assume that the vector fields in $\mathcal{F}$ are of class C^∞. In fact, this is a standard assumption in (continuous-time) nonlinear control, and it is justified by the fact that most relevant systems in mechanics satisfy this assumption. Then we look at the smallest Lie algebra of vector fields containing $\mathcal{F}$, which we denote by $\mathcal{L}(\mathcal{F})$. It turns out that the system is locally accessible at x if $\mathcal{L}(\mathcal{F})(x) = \{g(x) : g \in \mathcal{L}(\mathcal{F})\} = T_xM$. The converse is true only on an open and dense set in M. Equivalence holds for real-analytic vector fields.

Let us check this for linear systems: Here

$$\mathcal{F} = \{Ax + Bu \ : \ u \in U\}$$

and we impose the assumption that $0 \in \mathrm{int}U$. Then $f(x) = Ax + b_i$, $b_i = Be_i$, is an element of $\mathcal{L}(\mathcal{F})$ for $i = 1, \ldots, m$. Recall that in $\mathbb{R}^d$ the Lie bracket of vector fields $X, Y : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$[X, Y](x) = DY(x)X(x) - DX(x)Y(x).$$

Then we have

$$[b_i, Ax] = Ab_i \in \mathcal{L}(\mathcal{F}), \ [[Ax, b_i], Ax] = A^2 b_i \in \mathcal{L}(\mathcal{F}), \ldots$$

Going on in this way, we find that the vector fields

$$Ax, b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, \ldots, A^{d-1}b_1, \ldots, A^{d-1}b_m$$

are in $\mathcal{L}(\mathcal{F})$, and the linear span of those is closed under Lie brackets. Using Caley-Hamilton again, evaluating at $x = 0$ thus gives

$$\mathcal{L}(\mathcal{F})(0) = \left\langle \mathrm{im}B, \mathrm{im}AB, \mathrm{im}A^2B, \ldots, \mathrm{im}A^{d-1}B \right\rangle = \langle A | \mathrm{im}B \rangle$$

and we find that local accessibility at $x = 0$ is equivalent to Kalman's rank condition (but not necessarily at other points).

4 Control Sets

Looking at a linear system with bounded controls, we can ask whether there exist subsets of the state space where the system is completely controllable, since we cannot expect complete controllability on the whole state space.

We make the following definition: A set $D \subset M$ is called a *control set* if it satisfies the following properties:

(i) D is *controlled invariant*, i.e., for every $x \in D$ there is $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$.

(ii) Approximate controllability holds on D, i.e., for any $x, y \in D$ and any neighborhood N of y there are $u \in \mathcal{U}$ and $t \geq 0$ with $\varphi(t, x, u) \in N$.

(iii) D is maximal (w.r.t. set inclusion) with properties (i) and (ii).

As an example, consider the planar linear system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t), \quad u(t) \in U = [-1, 1].$$

The solutions are

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{t} x_0 \\ \mathrm{e}^{-t} y_0 \end{pmatrix} + \int_0^t u(s) \begin{pmatrix} \mathrm{e}^{t-s} \\ \mathrm{e}^{s-t} \end{pmatrix} \mathrm{d}s.$$

In this case, a control set is given by

$$D = (-1, 1) \times [-1, 1].$$

It can be seen that this is a control set by looking at the phase portraits for the constant controls $u_1 = 1$ and $u_2 = -1$. To see that the set is maximal, think of the phase portraits for the other constant controls, and notice that it is sufficient to look at piecewise constant controls. This example shows that in general a control set is neither open nor closed. Not all control sets are so ‘nice’.

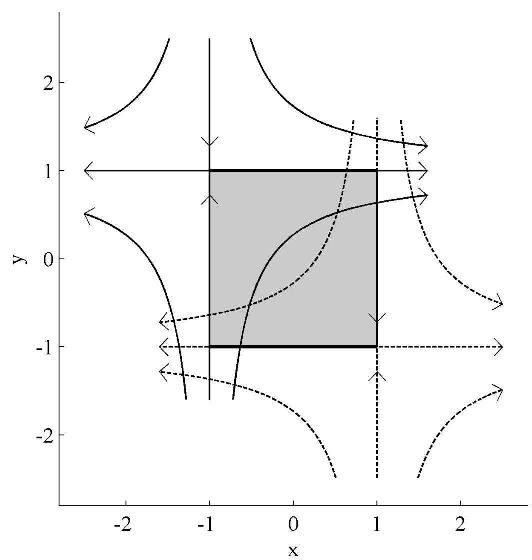

Figure 1: The control set D of the planar linear system

Here are two rather pathological examples:

(a) Consider on $M = \mathbb{R}$ the system

$$\dot{x}(t) = u(t), \quad u(t) \in U \subset \mathbb{R}.$$

If $U \subset (0, \infty)$, there is no control set. If $U = \{0\}$, there is a continuum of one-point control sets.

(b) Consider on $M = \mathbb{R}^2$ the system

$$\dot{x}(t) = \begin{pmatrix} 0 & u(t) \\ -u(t) & 0 \end{pmatrix} x(t), \quad u(t) \in U \subset \mathbb{R}.$$

If $U \neq \{0\}$, then all circles around 0 are control sets.

Different control sets are disjoint, since otherwise their union would satisfy properties (i) and (ii) of control sets. However, it is possible to have infinitely many control sets with nonempty interiors on a compact state space, and also disconnected control sets (with empty interior) are possible.

Under the assumption of local accessibility, control sets with nonempty interior have nice properties. First notice that if D is a maximal set of complete approximate controllability and $\mathrm{int}D \neq \emptyset$, then D is already a control set. Controlled invariance is satisfied, because for each $x \in D$ one can first steer to $\mathrm{int}D$ and then run back and forth between two disjoint open sets. Because of maximality, this is possible without leaving the set D. In general, a control set D with *nonempty interior* has the following properties, which all are easy to prove:

(a) There are no trajectories that leave D and return.

(b) If the system is locally accessible on $\mathrm{cl}\, D$, then $\mathrm{cl}\, D = \mathrm{cl}\,\mathrm{int}D$ and D is connected.

(c) If local accessibility holds on $\mathrm{int}D$, then for all $y \in \mathrm{int}D$

$$D = \mathrm{cl}\,\mathcal{O}^+(y) \cap \mathcal{O}^-(y).$$

In particular, on $\mathrm{int}D$ exact controllability holds.

Here

$$\mathcal{O}^+(x) = \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x), \quad \mathcal{O}^-(x) = \bigcup_{T>0} \mathcal{O}^-_{\leq T}(x)$$

are the positive and negative orbits of x, respectively.

To show (b), one uses that $\mathrm{cl}\,\mathcal{O}^+(x) = \mathrm{cl}\,\mathrm{int}\mathcal{O}^+(x)$. For a linear control system satisfying Kalman's rank condition and having a compact and convex control range U with $0 \in \mathrm{int}U$, a unique control set with nonempty interior, given by

$$D = \mathrm{cl}\,\mathcal{O}^+(0) \cap \mathcal{O}^-(0),$$

exists. It is bounded iff the matrix A is hyperbolic.

Under the assumption of local accessibility, the boundary of a control set D with nonempty interior is the disjoint union of three sets, the *entrance boundary*, the *exit boundary* and the *tangential boundary*. The entrance boundary is the set from which one can steer into the interior, the exit boundary is the set of points one can reach from the interior, and the tangential boundary is the complement of those two. The entrance and the exit boundaries are open relative to ∂D and the tangential boundary has empty interior in ∂D. The entrance boundary belongs to the control set, while the other parts of the boundary do not. There exist control sets whose complete boundary is equal to the entrance boundary (so-called *invariant control sets*) and also such, where the whole boundary is the exit boundary (one obtains such a control set from an invariant control set by time-reversal).

In the two-dimensional linear example, the entrance boundary is given by $(-1,1) \times \{-1,1\}$ and the exit boundary is $\{-1,1\} \times (-1,1)$. The tangential boundary consists of the four edges of $[-1,1] \times [-1,1]$.

A dynamical interpretation of control sets is possible for the class of control-affine systems. These are systems of the form

$$\begin{aligned} \dot{x}(t) &= f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \quad u \in \mathcal{U}, \\ \mathcal{U} &= \{u : \mathbb{R} \to \mathbb{R}^m \ : \ u \text{ is measurable with } u(t) \in U \text{ a.e.}\}, \end{aligned}$$

where $U \subset \mathbb{R}^m$ is a compact and convex set. Then we clearly have $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$. Since $L^\infty(\mathbb{R}, \mathbb{R}^m)$ can be identified with the dual space of $L^1(\mathbb{R}, \mathbb{R}^m)$, we can consider the (relative) weak*-topology on $\mathcal{U}$. This is the smallest topology such that the functionals

$$u \mapsto \int_{\mathbb{R}} \langle u(t), x(t) \rangle \mathrm{d}t, \quad L^\infty(\mathbb{R}, \mathbb{R}^m) \to \mathbb{R}, \quad x \in L^1(\mathbb{R}, \mathbb{R}^m),$$

are continuous. Then we can prove the following facts:

(a) The space $\mathcal{U}$ is a compact metrizable space (use Banach-Alaoglu).

(b) The control flow on $\mathcal{U} \times M$ is a continuous dynamical system.

(c) If $f_0, f_1, \ldots, f_m$ are of class $\mathcal{C}^k$, then so are the maps $\varphi_{t,u} := \varphi(t, \cdot, u) : M \to M$. Moreover, the map $(t, x, u) \mapsto \mathrm{D}^k \varphi_{t,u}(x)$ is continuous.

(d) The shift θ on $\mathcal{U}$ is chain transitive, topologically mixing, and has dense periodic points.

The proofs of (a) and (b) are pretty straightforward, using the definition of the weak*-topology and standard results from functional analysis, and the theorem of Arzelà-Ascoli in (b). It can easily be shown that the periodic functions are dense in $\mathcal{U}$, which implies chain transitivity.

Now we introduce a notion similar to control sets, using pseudo-trajectories (chains) instead of trajectories. Let d be a metric on M. Then a controlled (ε, T)-chain from x to y is given by $n \in \mathbb{N}$, points $x = x_0, \ldots, x_n = y$, controls

$u_0, \dots, u_{n-1}$ and times $t_0, \dots, t_{n-1} \geq T$ such that $d(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon$ for $i = 0, 1, \dots, n-1$. A set $E \subset M$ is called a *chain control set* if it satisfies the following three properties:

(i) E is full-time controlled invariant, i.e., for every $x \in E$ there is $u \in \mathcal{U}$ with $\varphi(\mathbb{R}, x, u) \subset E$.

(ii) For all $x, y \in E$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain from x to y (in M).

(iii) E is maximal with (i) and (ii).

Some (easily proved) properties of chain control sets are the following:

(a) Chain control sets are closed and pairwisely disjoint.

(b) Under the assumption of local accessibility, each control set with nonempty interior is contained in a chain control set (construct periodic trajectories for arbitrary initial values in $\mathrm{int} D$ and use compactness of $\mathcal{U}$ and continuity of φ to show that $\mathrm{cl}\, D$ is full-time controlled invariant).

(c) The lift of a chain control set E to $\mathcal{U} \times M$, given by

$$\mathcal{E} = \{(u, x) \in \mathcal{U} \times M \ : \ \varphi(\mathbb{R}, x, u) \subset E\},$$

is a maximal invariant chain transitive set of the control flow. Conversely, the projection of a maximal invariant chain transitive set of the control flow to M is a chain control set.

Furthermore, we have the result that under the assumption of hyperbolicity (to be made precise) a chain control set with nonempty interior is the closure of a control set, provided that local accessibility holds. In the proof of this fact a nonautonomous shadowing lemma is used (cf. [3]).

There is a similar result to (c) above for control sets. Their lifts to $\mathcal{U} \times M$ (defined in a similar but not the same way as above) are maximal sets of topological transitivity for the control flow.

In the discrete-time case the theory of control sets is more complicated, but we are not going to discuss this.

5 Linearization and Local Controllability

Consider a continuous-time nonlinear system with a differentiable right-hand side f. Assume that $f(x_0, u_0) = 0$ for some $(x_0, u_0) \in M \times U$ (an *equilibrium pair*). Then one would expect that controllability of the linearization at (x_0, u_0) yields local controllability of the nonlinear system in a neighborhood of (x_0, u_0) in some sense. This is true, and it holds in greater generality. Fix a trajectory

$\varphi(\cdot, x_0, u_0)$ for some $(x_0, u_0) \in M \times \mathcal{U}$. We consider this trajectory only on a time interval of the form $[0, \tau]$. Then it makes sense to define the linearization along the trajectory by

$$\dot{z}(t) = A(t)z(t) + B(t)\mu(t), \quad \mu \in L^\infty([0,\tau], \mathbb{R}^m), \tag{2}$$

with

$$A(t) = \frac{\partial f}{\partial x}(\varphi(t, x_0, u_0), u_0(t)), \quad B(t) = \frac{\partial f}{\partial u}(\varphi(t, x_0, u_0), u_0(t)).$$

We may assume that $M \subset \mathbb{R}^d$ and then $A(t)$ and $B(t)$ are matrices of dimensions $d \times d$ and $d \times m$, respectively. (In the general case, it is technically more complicated. For instance, one can use a Riemannian metric and covariant derivatives.) We write $\phi^{x_0,u_0}(\cdot, \lambda, \mu)$ for the unique solution of (2) with $\phi^{x_0,u_0}(0, \lambda, \mu) = \lambda \in T_{x_0}M$. Then the identity

$$\phi^{x_0,u_0}(t, \lambda, \mu) = (\mathrm{D}\varphi_t)_{(x_0,u_0)}(\lambda, \mu)$$

holds, where on the right-hand side we use the Fréchet derivative with respect to the control variable. Let $x_1 := \varphi(\tau, x_0, u_0)$. Then the linearization is called *controllable on* $[0, \tau]$ if for all $\lambda_1 \in T_{x_0}M$ and $\lambda_2 \in T_{x_1}M$ there exists $\mu \in L^\infty([0,\tau], \mathbb{R}^m)$ with $\phi^{x_0,u_0}(\tau, \lambda_1, \mu) = \lambda_2$. In this case, we say that the *controlled trajectory* $(\varphi(\cdot, x_0, u_0), u_0(\cdot))$ is *regular on* $[0, \tau]$.

In a neighborhood of the controlled trajectory $(\varphi(\cdot, x_0, u_0), u_0(\cdot))$ the linearization approximates the nonlinear system in the sense that for all $t, C > 0$ there exists a function ζ such that

$$\|\varphi(t, y, \mu) - \varphi(t, x_0, u_0) - \phi^{x_0,u_0}(t, y - x_0, \mu - u_0)\| \le \zeta(b)b,$$

where $\zeta(b) \to 0$ for $b \to 0$, and $\|y - x_0\| \le b$, $\|\mu - u_0\| \le Cb$.

It can be shown that regularity implies *local controllability* in the following sense: For every $\varepsilon > 0$ there exists $\delta > 0$ such that for any two points $y \in B_\delta(x_0)$ and $z \in B_\delta(x_1)$ there is $v \in L^\infty([0,\tau], \mathbb{R}^m)$ with $\varphi(\tau, y, v) = z$ and $d(\varphi(t, y, v), \varphi(t, x_0, u_0)) < \varepsilon$ for all $t \in [0, \tau]$.

This is proved as follows: Consider the map $\alpha(x, \nu) = \varphi(\tau, x, \nu)$ and note that $\alpha(x_0, u_0) = x_1$. This map is continuously differentiable and its partial derivative with respect to ν is the map $\phi^{x_0,u_0}(\tau, 0, \cdot)$. From the controllability assumption it follows that this linear map is onto. Then the Implicit Mapping Theorem for maps from normed spaces into finite-dimensional spaces can be applied and it yields a $\mathcal{C}^1$-mapping $j : B_{\varepsilon_1}(x_0) \times B_{\varepsilon_1}(x_1) \to L^\infty([0,\tau], \mathbb{R}^m)$ such that $\alpha(y, j(y, z)) = z$ for all (y, z) in the domain of j and such that $j(x_0, x_1) = u_0$. Given any $\varepsilon > 0$ pick $\delta > 0$ such that $d_\infty(\varphi(\cdot, x_0, u_0), \varphi(\cdot, y, j(y, z))) < \varepsilon$ whenever $y \in B_\delta(x_0)$ and $z \in B_\delta(x_1)$. By continuity of j and continuous dependence on initial conditions such δ exists. (Note that then also the control function $j(y, z)$ is close to u_0.)

The converse is in general not true: Local controllability can hold if the linearization is not controllable. Consider for $M = \mathbb{R}^2$ and $U = \mathbb{R}$ the system given by

$$\begin{aligned} \dot{x}_1 &= x_1 + \sin x_2 + x_1 \mathrm{e}^{x_2}, \\ \dot{x}_2 &= x_2^2 + u. \end{aligned}$$

Then $(x_1, x_2) = (0, 0)$ is an equilibrium for $u = 0$.[5] The linearization is

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since B is not an eigenvector of A, the smallest A-invariant subspace containing $\mathrm{im}B = \mathbb{R} \cdot B$ is $\mathbb{R}^2$, and hence the linearization is controllable. If we replace u by u^2 in the second equation, we find that $B = 0$, and hence the linearization is no longer controllable. In this case, also the nonlinear system is no longer locally controllable, since one cannot reach points of the form $(0, x_2)$ with $x_2 < 0$ from the initial state $(0, 0)$. However, if we replace u by u^3, we still have $B = 0$, but now the same control functions as in the first case are available for the nonlinear system and hence local controllability holds again.

References

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[5] Note that in the case of an equilibrium the time τ can be chosen arbitrarily.